From Microscopic Driver Models to Macroscopic PDEs in Ring Road Traffic Dynamics

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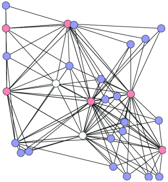


Motivations

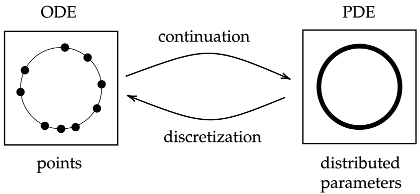
- **Stop-and-go waves**: Emerge from collective behavior.
- **Ring-road experiments**: Confirm oscillations [Treiterer 74].

Figure: Sugiyama et al 2008

Figure: Stern et al 2018

Modeling Approaches & Contribution

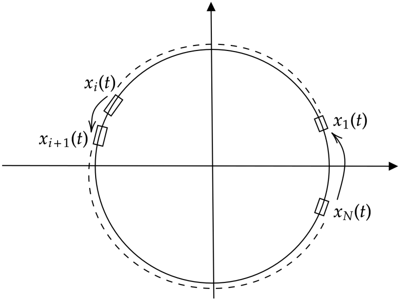
- **Microscopic (ODEs) models:**
  - OV-FTL: [Bando et al 1995, Cui et al 2017]
- **Macroscopic (PDEs):**
  - ARZ model [Aw and Rascle 2000, Zhang 2002]
- **Micro/Macro:**
  - Link ARZ and OV-FTL [Aw et al 2002]



Contribution:

- Systematic method
- Stability analysis
- Open control questions

Second order driver models on a ring



- Model (ODEs):

$$\ddot{x}_i = f(\underbrace{x_{i+1} - x_i}_{\text{interdistance}}, \underbrace{\dot{x}_{i+1} - \dot{x}_i}_{\text{intervelocity}}, \underbrace{\dot{x}_i}_{\text{velocity}}), \quad \forall i = 1, \cdots, N, \quad \forall t \geq 0$$

- N vehicles
- $x_i = x_i \bmod 2\pi, \quad x_i \in [0, 2\pi)$

OV-FTL model

Example: OV-FTL Model

$$\ddot{x}_i = \underbrace{a\frac{\dot{x}_{i+1} - \dot{x}_i}{(x_{i+1} - x_i)^2}}_{\text{Collision prevention}} + \underbrace{b\big(V(x_{i+1} - x_i) - \dot{x}_i\big)}_{\text{Desired velocity}}$$

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Key Features:

- a, b : control gains.
- $V(\cdot)$: Desired velocity (inspired from FD), decreasing with the inverse of the interdistance "density" (e.g., Greenshield's model).

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How to deduce a continuous model?

Continuation method–Base steps

Based on: [Nikitin, Canudas-de-Wit, Frasca, *TAC* 21]

1. See discrete set $(x_i)_{i=1,\cdots,N}$ as a continuum of vehicles described by a function $x(t,M) : \mathbb{R}_+ \times [0,N]$ with value in $[0,2\pi)$ such that $x(t,i) = x_i(t)$;

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4. Define the speed variable v and do Taylor approximations to obtain PDEs.

Continuation Process–details

Notations: ρ : density, v : velocity

Approximations:

$$x \approx x_i, \quad v \approx \dot{x}_i, \quad \partial_t v \approx \ddot{x}_i,$$

$$\rho \approx \frac{1}{x_{i+1} - x_i}, \quad \partial_x v \approx \dot{x}_{i+1} - \dot{x}_i, \ 1^{st}\text{-order}$$

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Interpretation Moskowitz function:

$$\rho(t,x) = \partial_x M \quad \text{and} \quad \varphi := v\rho = -\partial_t M$$

$$x_{i+1} - x_i \approx \frac{\partial x}{\partial M}, \quad \dot{x}_{i+1} - \dot{x}_i \approx \frac{\partial}{\partial M}\frac{\partial x}{\partial t}.$$

By consistency: $\partial_{tx} M = \partial_{xt} M$, leading to

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t v = f\left(\frac{1}{\rho}, \frac{\partial v}{\partial x}, v\right), \end{cases} \qquad \begin{cases} \rho(t,0) = \rho(t,2\pi), & \forall t \geq 0, \\ v(t,0) = v(t,2\pi), & \forall t \geq 0. \end{cases}$$

From Modeling to Stability Analysis

- We have shown how to derive a **macroscopic model** (PDE for the vehicle density) from a **microscopic car-following model** by passing to the continuum limit.
- This raises the following natural question:

Main Question

Are the **stability properties of equilibrium configurations** preserved when passing from the microscopic to the macroscopic model?

- The next section addresses this question by comparing the stability of equilibria in both frameworks.

Instability analysis OV-FTL (1)

Definition

A pair of real positive numbers (d^*, v^*) is called a *homogeneous equilibrium* of the OV-FTL model if, for all $i = 1, \ldots, N$,

$$x_{i+1} - x_i := d^*, \quad \dot{x}_{i+1} - \dot{x}_i = 0, \quad \text{and} \quad \dot{x}_i = v^*$$

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At a homogeneous equilibrium:

- the vehicles are evenly spaced by a distance d^*
- each vehicle moves at the constant speed v^*

Instability analysis OV-FTL (2)

Proposition (Cui et al 2017)

The homogeneous equilibrium points of the OV-FTL model is locally unstable if and only if the following inequality holds

$$a - \frac{2b}{(d^*)^2} - 2V'(d^*) < 0. \quad (1)$$

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$\Longrightarrow$ Stop and go phenomena

Macroscopic OV-FTL

Continuation method $\implies$ hyperbolic system:

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho \\ v\end{bmatrix} + \begin{bmatrix} v & \rho \\ 0 & -a\rho^2 \end{bmatrix} \frac{\partial}{\partial x}\begin{bmatrix}\rho \\ v\end{bmatrix} = \begin{bmatrix} 0 \\ b\left(V\left(\frac{1}{\rho}\right) - v\right)\end{bmatrix} \qquad (2)$$

with periodic boundaries.

- $\rho = 1/d^*$ and $v = v^*$ is an equilibrium point of the hyperbolic system.

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- If a enough small, instability of equilibrium points of macroscopic and microscopic points aligns.

Conclusion

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Next questions:

- How to use the continuation method to stabilize?