A Continuation-Based Control Strategy for Stabilizing Second-Order Macroscopic Traffic Flow on Circular Roads

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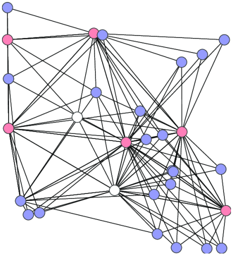


Motivations

- **Stop-and-go waves**: Emerge from collective behavior.
- **Confirmed by observation**: [Treiterer 74].
- **Ring-road experiments**:

Figure: Sugiyama et al. 2008

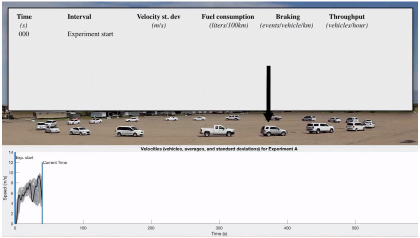
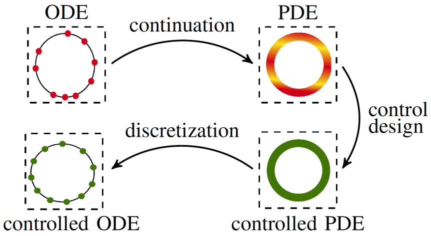


Figure: Stern et al. 2018

Control approaches & Contribution

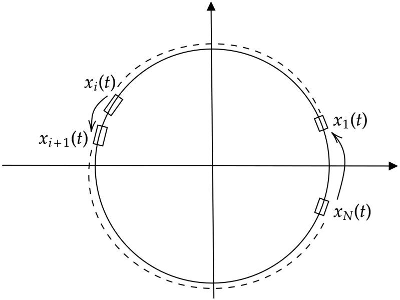
- **Microscopic (ODEs)–Single AV control:**
  - Local analysis (Ring) [Cui et al 17, Delle Monache et al 19, Hayat et al 23].
- **Macroscopic (PDEs)-Boundary control**
  - Closed-loop control (Line networks), [Yu & Krstic 18].
  - Open-loop control (Ring) [Goatin 24].
- **Micro/Macro–All AVs control**
  - 1^{st}-order mean field (Ring) [Maffettone et al 23].



Contribution:

- ODE in a ring
- Transform to a PDE
- Controlled PDE
- Discretize back −− > Controlled ODE

Second order models on a ring



- Model (ODEs):

$$\ddot{x}_i(t) = u_i(t), \quad \forall i = 1, \cdots, N, \quad \forall t \geq 0$$

- N autonomous vehicles
- $x_i = x_i \bmod 2\pi, \quad x_i \in [0, 2\pi)$

Typical Driver Laws

General form:

$$u_i(t) = f(\underbrace{x_{i+1} - x_i}_{\text{interdistance}}, \underbrace{\dot{x}_{i+1} - \dot{x}_i}_{\text{intervelocity}}, \underbrace{\dot{x}_i}_{\text{velocity}})$$

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Example: OV-FTL Model

$$\ddot{x}_i = \underbrace{K_p \frac{\dot{x}_{i+1} - \dot{x}_i}{(x_{i+1} - x_i)^2}}_{\text{Collision prevention}} + \underbrace{K_v (V(x_{i+1} - x_i) - \dot{x}_i)}_{\text{Desired velocity}}$$

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Key Features:

- K_p, K_v : control gains.
- $V(\cdot)$: Desired velocity (inspired from FD), decreasing with the inverse of the interdistance "density" (e.g., Greenshield's model).

Uniform Equilibrium points

Definition

A pair of real positive numbers (d^*, v^*) is called an uniform equilibrium point if, for all $i = 1, \cdots, N$,

$$x_{i+1} - x_i := d^*, \quad \dot{x}_{i+1} - \dot{x}_i = 0 \quad \text{and} \quad \dot{x}_i = v^*$$

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At an uniform equilibrium point:

- d^* is imposed by the circular topology and equals $2\pi/N$.
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Some functions $f(\cdot, \cdot, \cdot)$ may lead to instable uniform equilibrium points

Continuation method–Base steeps

Based on: [Nikitin, Canudas-de-Wit, Frasca, *TAC* 21]

1. See discrete set $(x_i)_{i=1,\cdots,N}$ as a continuum of vehicles described by a function $x(t,M) : \mathbb{R}_+ \times [0,N]$ with value in $[0,2\pi)$ such that $x(t,i) = x_i(t)$;

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4. Define the speed variable v and do Taylor approximations to obtain PDEs.

Continuation Process–details

Notations: ρ : density, v : velocity, u : control

Approximations through Moskowitz functions leads to:

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t v = u, \end{cases} \qquad \begin{cases} \rho(t,0) = \rho(t,2\pi), & \forall t \geq 0, \\ v(t,0) = v(t,2\pi), & \forall t \geq 0. \end{cases}$$

Feedback law

- The time derivative of the first transport equation, leads to:

$$\begin{cases} \partial_{tt}\rho + \partial_x(u\rho - v\partial_x(\rho v)) = 0, & \text{(1a)} \\ \partial_t v = u \qquad\qquad\qquad\qquad , & \text{(1b)} \end{cases}$$

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- Feedback law: $\alpha, \beta > 0$

$$u = \frac{1}{\rho}\left(\underbrace{v\partial_x(\rho v)}_{\substack{\text{cancel } v \\ \text{in (1a)}}} + \underbrace{\alpha(\overline{\rho}\,\overline{v} - \rho v)}_{\substack{\text{introduce } \alpha\partial_t\tilde{\rho} \\ \text{in (1a)}}} - \underbrace{\beta\partial_x\tilde{\rho}}_{\substack{\text{introduce } \beta\partial_{xx}\tilde{\rho} \\ \text{in (1a)}}} \right)$$

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Convergence error equations

Theorem

Under the feedback control u *defined in* (12), *the equations errors* $\tilde{\rho} = \rho - \overline{\rho}$ *and* $\tilde{v} = v - \overline{v}$ *converge uniformly exponentially to zero, i.e., there exist constants* $a, b, c, d > 0$ *such that*

$$\|\tilde{\rho}(t, \cdot)\|_\infty \leq a e^{-bt},$$
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Strategy of Proof:

1. Fourier analysis $\tilde{\rho}$;
2. Cascade structure for $\tilde{v}$.

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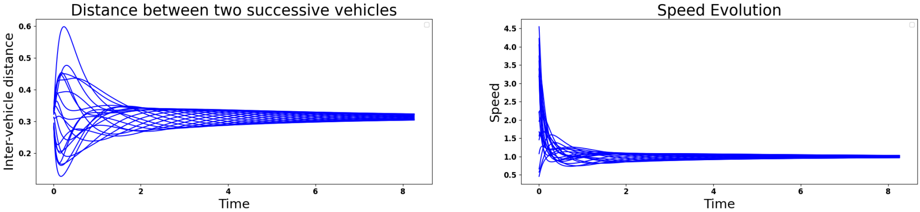
Selected microscopic control:

$$u_i = (x_{i+1} - x_i) \left[\frac{\dot{x}_{i+1} - \dot{x}_i}{(x_{i+1} - x_i)^2} + \alpha \left(\frac{v^*}{d^*} - \frac{\dot{x}_i}{x_{i+1} - x_i} \right) \right.$$
$$\left. - \beta \left(\frac{1}{x_{i+1} - x_i} - \frac{1}{x_i - x_{i-1}} \right) \right]$$

Numerical simulations

$\alpha = 5$, $\beta = 1$, $v^* = 1$, $d^* = 2\pi/N \approx 0.32$ and $N = 20$.
The initial positions and speeds: random



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Next questions :

- How to adapt the method in the case where is driver laws?
- How much autonomous vehicles do we have to control?